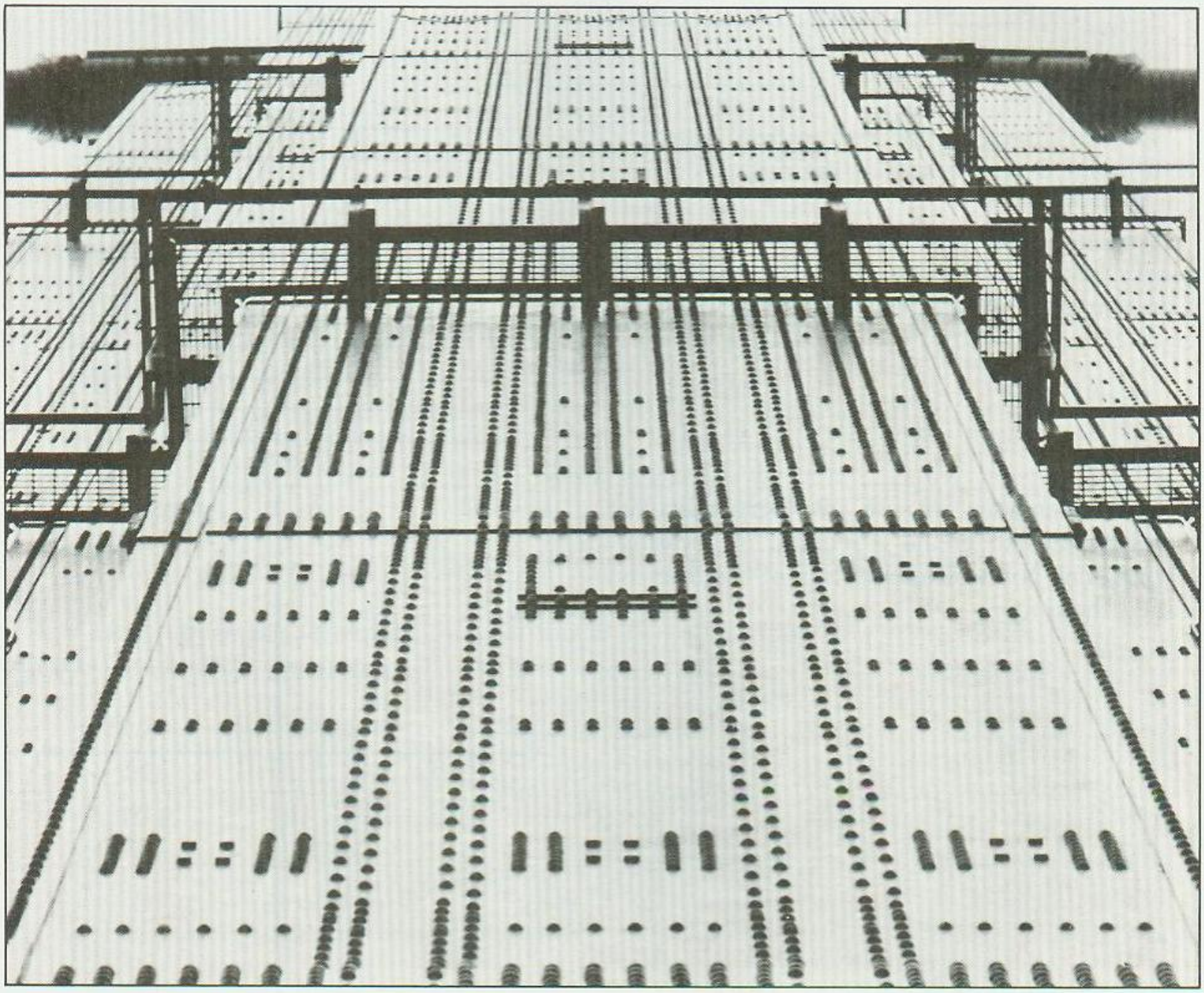
CHAPTER 14

14–1 Cartesian Coordinate System 504
14–2 Midpoint of a Segment 508
14–3 Slope of a Line 512
14–4 Slopes of Perpendicular and Parallel Lines 516
14–5 The Distance Formula 520
14–6 The Equation of a Line 524
14–7 The Equation of a Circle 528
14–8 Using Coordinates to Prove Theorems 532
14–9 Transformations and Coordinate Geometry 536
Important Ideas 538 *Review* 539 *Test* 540
Cumulative Review (Chapters 11–14) 541

Coordinate Geometry



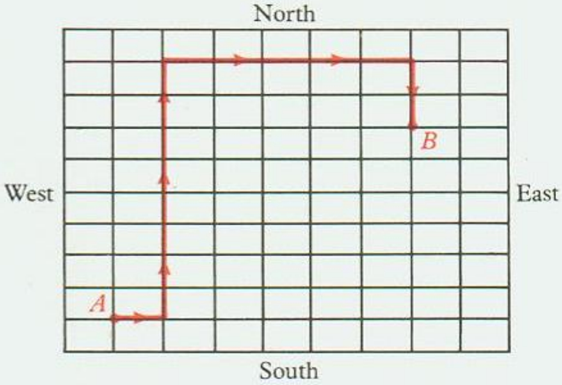
14–1 Cartesian Coordinate System

How would you tell someone to get from one point to another? When giving directions we often say to go a certain distance in one direction and then go a certain distance in a different direction.

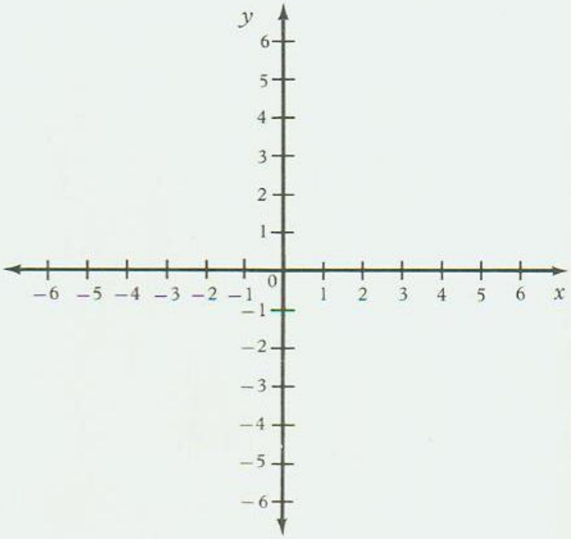
To give directions for getting from point A to point B on the grid at the right, you could say:

"Go one block east, eight blocks north, five blocks east, and then two blocks south."

Can you give two simpler ways to get from A to B?

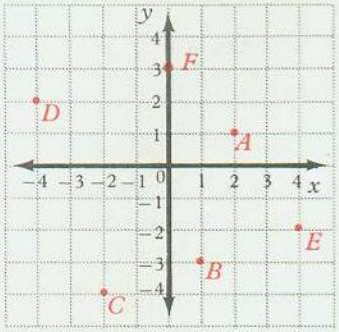


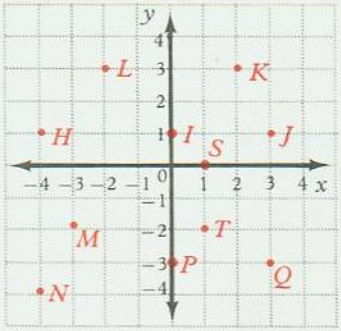
In mathematics we use two perpendicular number lines to create a method of locating points. The point of intersection of the lines is called the *origin.*



A pair of numbers called *coordinates* name the location of each point. Point A, below, is "over 2" and "up 1" from the origin. We say that A has coordinates (2, 1). The first number is called the x-coordinate and the second is the y-coordinate. In general, a point is represented by the coordinates (x, y). We will use the notation $P(x, y)$ to represent the point P with coordinates (x, y). This method of determining points is called the *Cartesian coordinate system* and is named after the famous mathematician René Descartes.

Study the examples below.



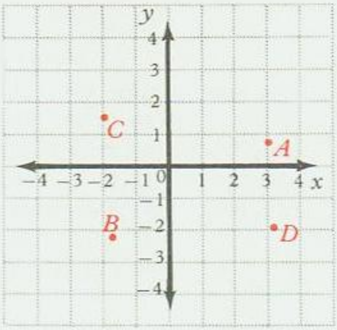


Point	*Coordinates*
A	$(2, 1)$
B	$(1, -3)$
C	$(-2, -4)$
D	?
E	?
F	?

Point	*Coordinates*
J	$(3, 1)$
H	$(-4, 1)$
I	$(0, 1)$
?	$(-3, -2)$
?	$(1, 0)$
?	$(0, -3)$

The examples below show that the coordinates do not have to be integers. Study each example.

Point	*Coordinates*
A	$(3, \frac{2}{3})$
B	$(-1\frac{4}{5}, -\frac{7}{3})$
C	$(-2, \sqrt{2})$
D	$(\pi, -2)$



EXERCISES

A.

In exercises 1–4, plot the following sets of points on graph paper. Use a different set of axes for each exercise.

1. $(-4, 2), (6, -1), (-5, -4)$, and $(-3, -2)$

2. $(6, 2), (-2\frac{1}{2}, \frac{2}{3}), (-5, \frac{8}{3})$, and $(-\sqrt{6}, \sqrt{3})$

3. $(-4, 0), (5, 0), (0, 0)$, and $(2\frac{1}{2}, 0)$

4. $(0, -2\frac{1}{2}), (0, 6), (0, -3)$, and $(0, \sqrt{2})$

In exercises 5–8, draw lines that contain the following pairs of points.

5. $(0, 0)$ and $(-5, -5)$

6. $(4, 0)$ and $(0, -3)$

7. $(-4, 2)$ and $(-2, -6)$

8. $(5, -3)$ and $(4, -3)$

In exercises 9–11, draw triangles with the following coordinates as vertices. Tell whether the triangles are acute, right, or obtuse.

9. $(-2, 3), (4, 1)$, and $(7, -2)$

10. $(3, 3), (-4, 0)$, and $(4, 6)$

11. $(-4, 2), (-4, -3)$, and $(3, -3)$

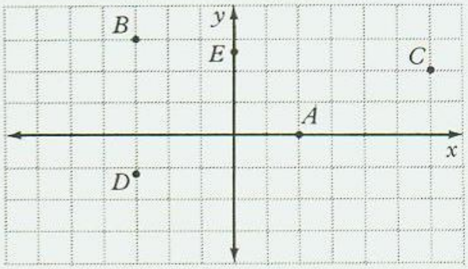
In exercises 12–14, draw quadrilaterals with the following coordinates as vertices.

12. $(-5, -3), (-1, -3), (-5, 1)$, and $(-1, 1)$

13. $(0, 6), (6, 0), (0, -6)$, and $(-6, 0)$

14. $(-1, 5), (3, 2), (2, -5)$, and $(-4, -1)$

15. Given the graph on the right, name the coordinates of points A, B, C, D, and E. In some cases you may have to estimate the coordinates.



Activity

On a coordinate grid start at (0, 0) and draw a continuous sequence of segments going from point to point. Read down each column.

WHAT OBJECT IS REPRESENTED?

Start → (0, 0)
(10, 0)
(12, 2)
(12, 8)
(8, 14)
(10, 17)
(10, 22)
(6, 24)
(6, 35)
(7, 35)
(6, 35)
(6, 37)
(7, 37)
(6, 37)
(6, 39)
(7, 39)
(6, 39)
(6, 42)
(4, 42)
(4, 39)
(3, 39)
(4, 39)
(4, 37)
(3, 37)
(4, 37)
(4, 35)
(3, 35)
(4, 35)
(4, 24)
(0, 22)
(0, 17)
(2, 14)
(−2, 8)
(−2, 2)
(0, 0) ← Stop

B.

In exercises 16–18, the coordinates of three vertices of a rectangle are given. Find the coordinates of the fourth vertex. (*Hint:* You may want to graph the three coordinates.)

16. $(0, 0)$, $(-4, 0)$, and $(0, -2)$

17. $(-4, 3)$, $(-4, -1)$, and $(5, -1)$

18. $(1, -3)$, $(1, 5)$, and $(4, 5)$

In exercises 19–21, the coordinates of three vertices of a square are given. Find the coordinates of the fourth vertex.

19. $(0, 0)$, $(3, 0)$, and $(0, -3)$

20. $(-1, 2)$, $(2, 2)$, and $(2, -1)$

21. $(-3, 2)$, $(-3, 5)$, and $(0, 5)$

In exercises 22 and 23, the coordinates of three vertices of a parallelogram are given. Find the coordinates of the fourth vertex. (There is more than one correct answer.)

22. $(0, 0)$, $(4, 4)$, and $(6, 4)$

23. $(1, 1)$, $(4, 1)$, and $(0, -1)$

24. Plot several points in which the product of the coordinates equals 12. Draw a smooth line through the points.

25. Plot several points in which the y-coordinate is twice the x-coordinate. Draw a line through the points.

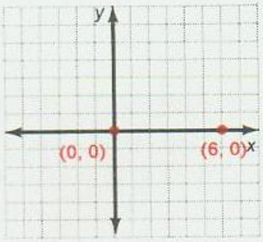
26. Find a third point that lies on the line through the two points $(3, -2)$ and $(-1, 2)$.

27. Graph the points $(2, 0)$ and $(0, 4)$ and draw a line through the points. If the coordinates $(0, 0)$ and $(3, n)$ determine a line parallel to the first line, what is the value of n?

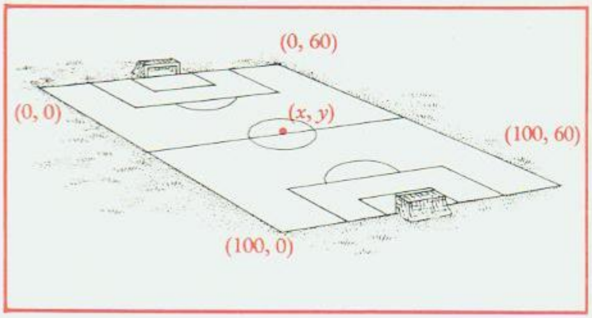
PROBLEM SOLVING

Two vertices of a figure are $(0, 0)$ and $(6, 0)$.

1. What are the coordinates of the third vertex if the figure is an equilateral triangle? (There are two solutions.)
2. What are the coordinates of the other two vertices if the figure is a square? (There are three solutions.)
3. What are the coordinates of the other two vertices if the figure is a parallelogram with height 4?



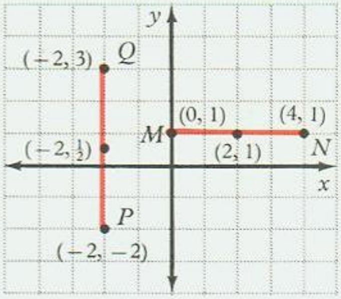
14–2 Midpoint of a Segment



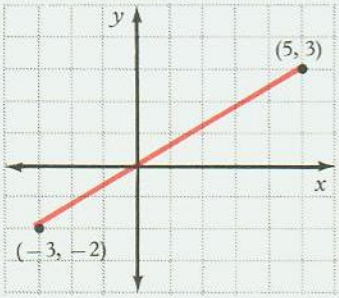
A soccer field is usually about 100 yards long and 60 yards wide. Suppose coordinates are assigned to the corners of the soccer field as shown. What are the coordinates of the center of the field?

Observe the midpoints of the segments at the right.

Observe that for the horizontal segment $\overline{MN}$, the x-coordinate of the midpoint is one half the sum of the x-coordinates of the endpoints. For the vertical segment $\overline{PQ}$, the y-coordinate of the midpoint is one half the sum of the y-coordinates of the endpoints.



This same idea can be used for other line segments. Let (x_1, y_1) be the coordinates of one endpoint and (x_2, y_2) be the coordinates of the other endpoint in the figure below.



$(x_1, y_1) = (-3, -2); (x_2, y_2) = (5, 3)$

$$\frac{x_1 + x_2}{2} = \frac{-3 + 5}{2} = 1$$

$$\frac{y_1 + y_2}{2} = \frac{-2 + 3}{2} = \frac{1}{2}$$

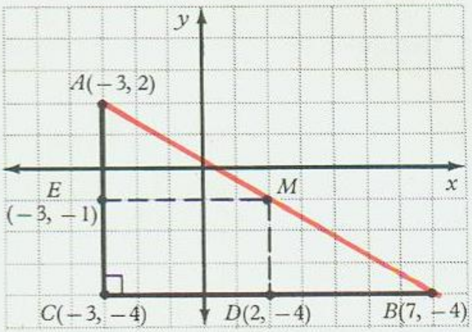
The coordinates of the midpoint are $(1, \frac{1}{2})$.

This example suggests the following theorem.

Theorem 14–1 If the coordinates of the endpoints of the segment $\overline{P_1P_2}$ are (x_1, y_1) and (x_2, y_2), then the coordinates of the midpoint of $\overline{P_1P_2}$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

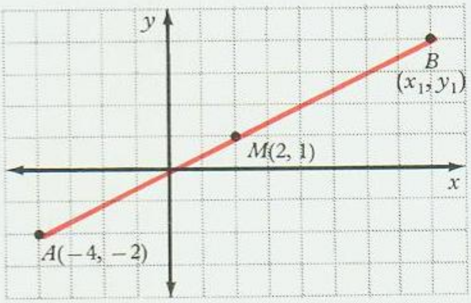
The following questions suggest how we might prove Theorem 14-1. The proof is left as an exercise.

1. Why are the coordinates of $C\ (-3, -4)$?
2. Why are the coordinates of E, the midpoint of $\overline{AC}$, $(-3, -1)$?
3. Why are the coordinates of D, the midpoint of $\overline{BC}$, $(2, -4)$?
4. Using questions 2 and 3, why are the coordinates of M, the midpoint of $\overline{AB}$, $(2, -1)$?



Note that if the coordinates of A are (x_1, y_1) or $(-3, 2)$ and the coordinates of B are (x_2, y_2) or $(7, -4)$, then $\frac{x_1 + x_2}{2} = \frac{-3 + 7}{2} = 2$ and $\frac{y_1 + y_2}{2} = \frac{2 + (-4)}{2} = -1$. Therefore, the coordinates of M are $(2, -1)$.

Example



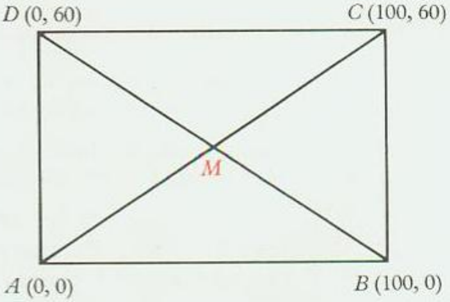
If M is the midpoint of $\overline{AB}$, where $(-4, -2)$ are the coordinates of A and $(2, 1)$ are the coordinates of M, find the coordinates of B.

Let (x_1, y_1) be the coordinates of B. By Theorem 14-1,

$$2 = \frac{-4 + x_1}{2} \quad \text{and} \quad 1 = \frac{-2 + y_1}{2}$$

Solving for x_1 and y_1, we obtain $x_1 = 8$ and $y_1 = 4$. The coordinates of B are $(8, 4)$.

APPLICATION



To find the coordinates of the center of the soccer field we must find the midpoint of $\overline{AC}$.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{0 + 100}{2}, \frac{0 + 60}{2}\right)$$

The coordinates of M are $(50, 30)$.

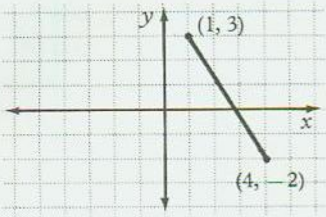
Would M have the same coordinates if we had found the midpoint of $\overline{BD}$?

EXERCISES

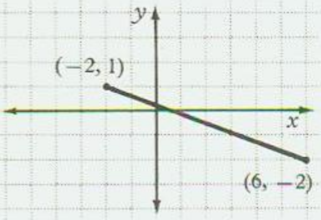
A.

1. Determine the midpoints of the following segments.

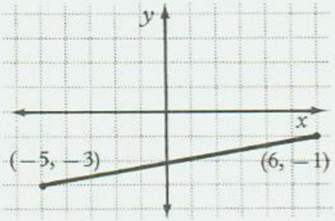
a.



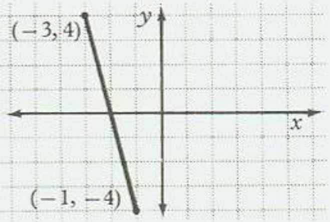
b.



c.



d.

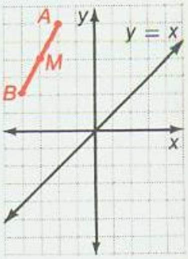


In exercises 2–7, determine the midpoints of segments whose endpoints have the following coordinates.

2. $(-3, 2)$ and $(7, 10)$
3. $(-7, 4)$ and $(9, -4)$
4. $(-1, -2)$ and $(5, 4)$
5. $(-3, -5)$ and $(-7, -2)$
6. $(-4, 2)$ and $(5, -3)$
7. $(7, -3)$ and $(0, 0)$
8. Graph the points $A(-2, 5)$, $B(6, 1)$, and $C(-4, -3)$ and draw $\triangle ABC$. Find the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.
9. Graph the points $A(1, 6)$, $B(6, 2)$, $C(8, -3)$, and $D(-5, 2)$. Find the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Draw a quadrilateral by connecting the midpoints. What kind of quadrilateral is it?

Activity

Suppose the coordinates of A are $(-2, 6)$ and the coordinates of B are $(-4, 2)$. By Theorem 14–1 the coordinates of midpoint M are $(-3, 4)$.



a. Use tracing paper or a MIRA to find the reflection of $\overline{AB}$ across the y-axis. Compare the coordinates of the midpoint of the reflected segment with the coordinates of M.

b. Reflect $\overline{AB}$ across the x-axis. Compare the coordinates of the midpoint of the reflected segment with the coordinates of M.

c. Make the same comparison when the line $y = x$ is used as a line of reflection.

B.

In exercises 10–14, M is the midpoint of $\overline{AB}$. The coordinates of two of the points are given. Find the coordinates of the third point.

10. $A(1, 1)$, $M(5, 1)$, find B.

11. $A(-2, -6)$, $M(-2, 1)$, find B.

12. $A(-5, -3)$, $M(2, 1)$, find B.

13. $M(0, 0)$, $B(-4, 3)$, find A.

14. $M(0, 0)$, $B(1, 6)$, find A.

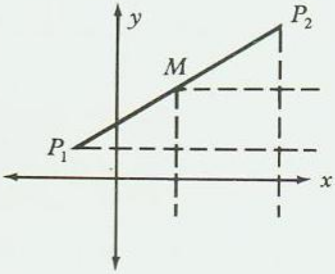
15. Given the triangle ABC with vertices $A(-4, -3)$, $B(8, 0)$, and $C(6, 12)$. A line is drawn parallel to base $\overline{AB}$ and bisecting $\overline{AC}$. Find the coordinates of the point where the line intersects $\overline{BC}$.

16. Suppose the coordinates of A and B are $(-4, 6)$ and $(6, -2)$. Find the coordinates of X such that $AX = \frac{1}{4}AB$.

17. Suppose the coordinates of A and B are $(-7, -2)$ and $(5, -1)$. Find the coordinates of a point C on $\overleftrightarrow{AB}$ such that $AC = \frac{1}{2}AB$.

C.

18. Prove Theorem 14–1 using the following information.
Given: $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_1M = MP_2$
Prove: The coordinates of M are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
(*Hint:* Draw the auxiliary lines that are indicated in the diagram.)

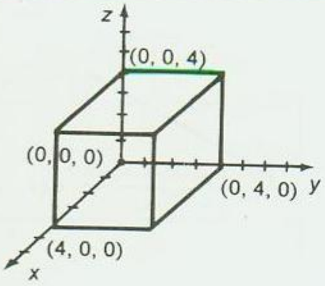


19. Consider the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Find the coordinates of the points that trisect $\overline{P_1P_2}$.

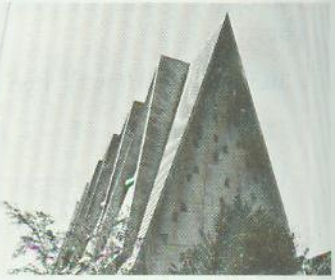
PROBLEM SOLVING

Consider a three-dimensional coordinate system with an *x*-axis, *y*-axis, and a *z*-axis. A cube with each edge of 4 units is placed as shown in the diagram.

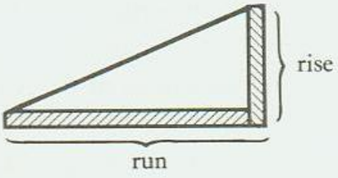
What are the coordinates (x, y, z) of the center of the cube?



14–3 Slope of a Line

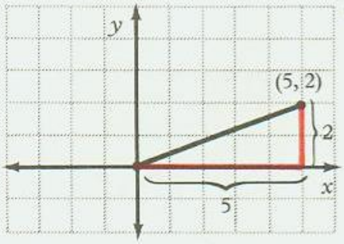


What is the slope or pitch of the roofs?

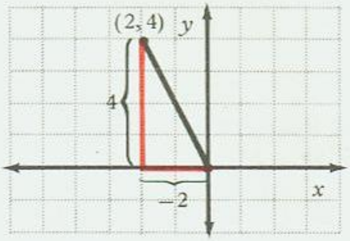


The slope of a line or a segment can be thought of as the ratio $\frac{\text{rise}}{\text{run}}$ as shown.

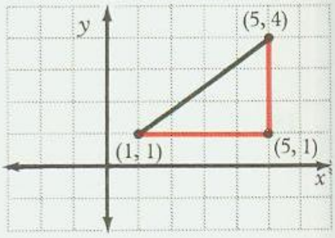
The letter m is used to designate slope. Study the following examples.



$$m = \frac{2}{5}$$

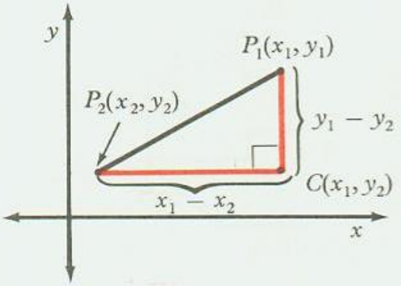


$$m = \frac{4}{-2} = -\frac{2}{1}$$



$$m = \frac{3}{4}$$

The definition of slope given below involves the Cartesian coordinate system.



The slope of a line is determined by the change in the vertical distance $(y_1 - y_2)$ divided by the change in the horizontal distance $(x_1 - x_2)$.

Definition 14-1

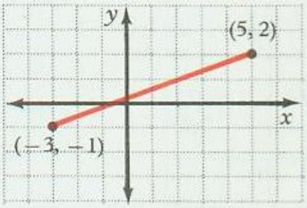
If P_1 and P_2 have coordinates (x_1, y_1) and (x_2, y_2) respectively, then the **slope** m of $\overline{P_1P_2}$ is:

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

provided $x_1 - x_2 \neq 0$.

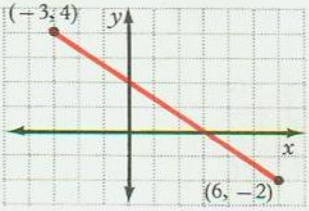
The examples below show that the slope of a line can be positive, negative, zero, or undefined.

Example 1 Positive slope



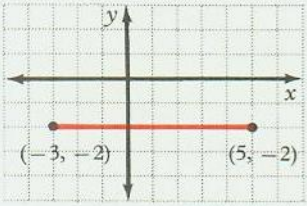
$$m = \frac{2-(-1)}{5-(-3)} = \frac{3}{8}$$

Example 2 Negative slope



$$m = \frac{4-(-2)}{-3-6} = -\frac{2}{3}$$

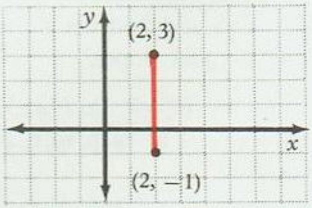
Example 3 Zero slope



$$m = \frac{-2-(-2)}{5-(-3)} = \frac{0}{8} = 0$$

The slope of a line parallel to the x-axis is zero.

Example 4 Undefined slope

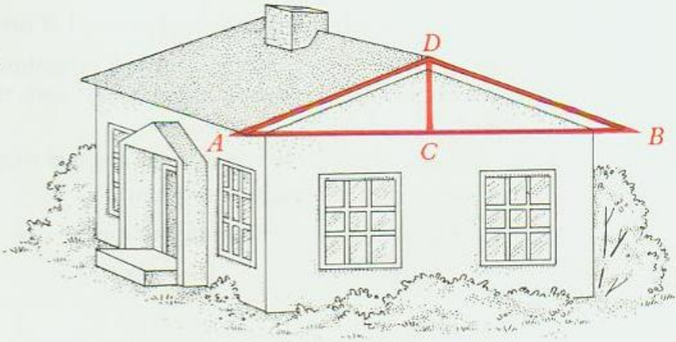


$$m = \frac{3-(-1)}{2-2} = \frac{4}{0}$$

Thus, the slope is undefined. Why? The slope of a line parallel to the y-axis is undefined.

APPLICATION

If this house is 30 feet wide and the rise on the roof is 6 feet, what is the slope of the roof?



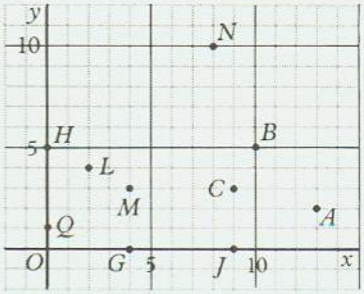
If $AB = 30$, then $AC = 15$.
Since $DC = 6$, then the

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{6}{15}.$$

Hence, the slope $= \frac{2}{5}$.

EXERCISES

A.



1. Determine the coordinates of each point and find the slope of the following segments.

   a. $\overline{AB}$ b. $\overline{HN}$ c. $\overline{AC}$

   d. $\overline{MC}$ e. $\overline{GJ}$ f. $\overline{LQ}$

   g. $\overline{HG}$ h. $\overline{MB}$ i. $\overline{MG}$

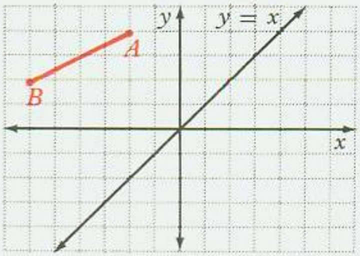
In exercises 2–7, find the slope of a line that contains the given points.

2. $(0, 0), (4, 6)$
3. $(3, 5), (9, 2)$
4. $(-2, 5), (6, -3)$
5. $(-2, 10), (-6, -4)$
6. $(10, 2), (-2, 2)$
7. $(5, 0), (0, -5)$
8. The vertices of a parallelogram are $R(1, 4)$, $S(3, 2)$, $T(4, 6)$, and $U(2, 8)$. Graph the points and find the slopes of each of the sides. Which sides have equal slopes?
9. The vertices of a rectangle are $E(-2, 3)$, $F(4, 3)$, $G(4, -1)$, and $H(-2, -1)$. Find the slopes of each side. For which two sides are the slopes undefined?

B.

10. What is the slope of a line if it crosses the x-axis at 6 and the y-axis at -2?
11. The vertices of a triangle are $A(4, 6)$, $B(-1, 2)$, and $C(2, -4)$. Find the slope of each side.

Activity



Suppose the coordinates of A are $(-2, 4)$ and B are $(-6, 2)$.

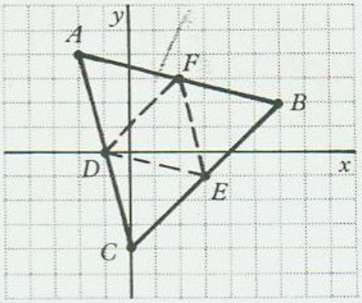
a. Use tracing paper or a MIRA to find the reflection of $\overline{AB}$ about the y-axis. Compare the slope of $\overline{AB}$ with the slope of the reflected segment.

b. Reflect $\overline{AB}$ about the x-axis and compare the slope of $\overline{AB}$ with the slope of the reflected segment.

c. Make the same comparison when the line $y = x$ is used as the line of reflection.

12. Given $\triangle ABC$ with $A(-2, 4)$, $B(6, 2)$, and $C(0, -4)$.

- **a.** Find the coordinates of the midpoints D, E, and F.
- **b.** Determine the slopes of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.
- **c.** Determine the slopes of $\overline{DE}$, $\overline{DF}$, and $\overline{FE}$.
- **d.** What do you observe about the slopes you found in **b** and **c** above?



13. The vertices of a triangle are $X(11, 0)$, $Y(-5, 4)$, and $Z(3, 4)$.

- **a.** Find the slopes of its sides.
- **b.** Find the slopes of its medians. (Use Theorem 14–1 to find the midpoints of the sides.)

14. A line with a slope of -3 crosses the x-axis at $(8, 0)$. At what point does it cross the y-axis?

15. What is the slope of a line if the x-coordinate is always twice the y-coordinate?

C.

16. $ABCD$ is a quadrilateral with vertices $A(a, b)$, $B(c, b)$, $C(c + d, e)$ and $D(a + d, e)$. Find the slope of each side.

17. $\triangle ABC$ has vertices $B(-6, -3)$ and $C(8, -4)$. Slope of $\overline{AB} = \frac{1}{2}$ and slope of $\overline{AC} = -2$. Find the coordinates of point A.

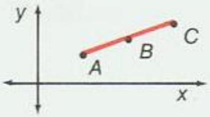
18. A line with a slope of -1 contains the point $(5, -2)$. Solve for x, if the line also contains the point $(x, 8)$.

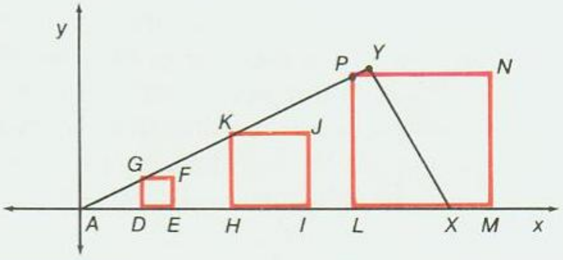
PROBLEM SOLVING

Three points A, B, and C are collinear if the slope of $\overline{AB}$ equals the slope of $\overline{BC}$. Use this fact to solve the following problem.

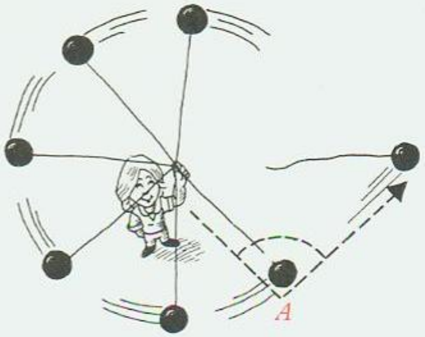
Given: $\triangle AXY$ with $A(0, 0)$, slope of $\overline{AY} = \frac{1}{2}$. $DEFG$, $HIJK$, and $LMNP$ are squares with these coordinates: $D(4, 0)$, $H(10, 0)$, and $L(18, 0)$.

Prove: F, J, and N are collinear.



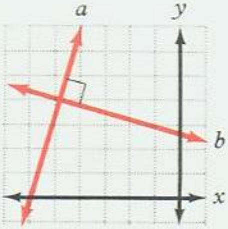


14–4 Slopes of Perpendicular and Parallel Lines

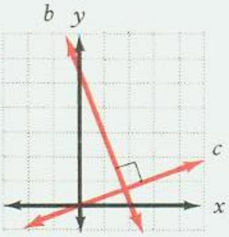


Suppose a ball on a string is twirled in a counterclockwise direction. If the ball is released at point A, what is the slope of the path that it follows? This question is examined at the bottom of the page.

First we shall consider the slopes of perpendicular lines. The lines shown are perpendicular. Determine the slope of each pair of lines.



Observe that the slope of $a = 4$, slope of $b = -\frac{1}{4}$, and $4 \cdot -\frac{1}{4} = -1$.



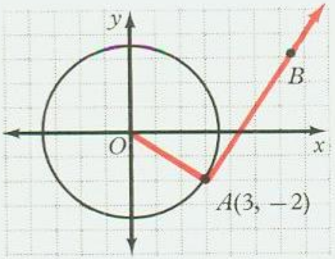
Observe that the slope of $b = -\frac{5}{2}$, slope of $c = \frac{2}{5}$, and $-\frac{5}{2} \cdot \frac{2}{5} = -1$.

These observations suggest the following theorem.

Theorem 14–2 The product of the slopes of two perpendicular lines is -1.

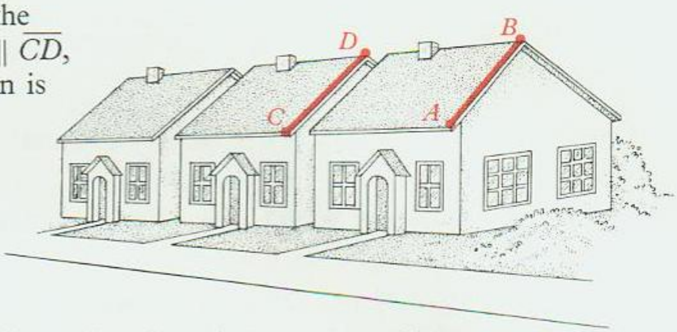
Theorem 14–2 is true only if neither line is parallel to the y-axis. Why?

APPLICATION

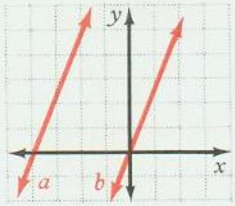


To solve the problem at the top of the page, assign a coordinate system. If the coordinates of A are $(3, -2)$, the slope of $\overline{OA}$ is $-\frac{2}{3}$. Surprisingly, the laws of science show that the ball will always move in a direction $\overrightarrow{AB}$ that is tangent to the circle. So $\overline{OA} \perp \overline{AB}$. By Theorem 14–2, the product of the slopes of these two segments is -1. So $-\frac{2}{3} \cdot$ slope of $\overline{AB} = -1$, or the slope of $\overline{AB} = \frac{3}{2}$.

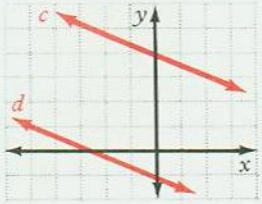
Consider the slopes of the roofs at the right. If the slope of $\overline{AB}$ is $\frac{2}{3}$ and $\overline{AB} \parallel \overline{CD}$, what is the slope of $\overline{CD}$? This question is answered at the bottom of the page.



Now consider the slopes of parallel lines. The lines below are parallel. Determine the slopes of each pair of lines.



Observe that the slope of $a = \frac{3}{1}$, slope of $b = \frac{3}{1}$, and slope of a = slope of b.



Observe that the slope of $c = -\frac{2}{5}$, slope of $d = -\frac{2}{5}$, and slope of c = slope of d.

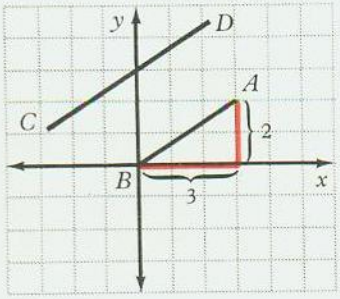
These observations suggest the following theorem.

Theorem 14–3 The slopes of two parallel lines are equal.

Theorem 14–3 is true only if neither line is parallel to the y-axis. Why?

APPLICATION

To answer the question at the top of the page assign a coordinate system. Then the slope of $\overline{AB} = \frac{2}{3}$. Theorem 14–3 tells us that the slope of $\overline{AB}$ equals the slope of $\overline{CD}$. Hence, the slope of $\overline{CD}$ equals $\frac{2}{3}$.



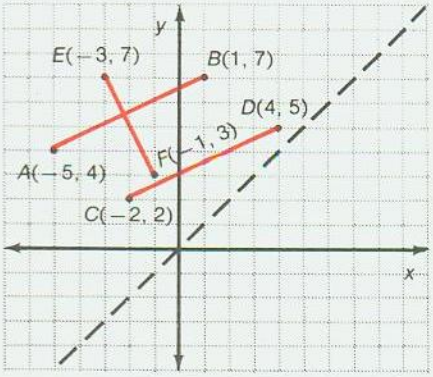
EXERCISES

A.

1. Consider the points $A(3, 5)$, $B(7, -1)$, $C(-4, 4)$, and $D(0, -2)$. Is $\overline{AB} \parallel \overline{CD}$?
2. Consider the points $A(3, 5)$, $B(7, -1)$, $C(0, 0)$, and $D(12, 8)$. Is $\overline{AB} \perp \overline{CD}$?
3. Show that $(3, 9)$, $(7, 5)$, $(4, -1)$, and $(0, 3)$ are the coordinates of the vertices of a parallelogram. (Use Theorem 14–3).
4. Show that $(1, 2)$, $(3, 1)$, $(0, -4)$, $(-2, -3)$ are the coordinates of the vertices of a parallelogram.
5. Show that $(1, -3)$, $(4, 5)$, $(-3, 7)$, and$(-6, -1)$ are the coordinates of the vertices of a parallelogram.
6. Show that $(4, 6)$, $(5, 1)$, and $(2, 4)$ are the coordinates of the vertices of a right triangle. (Use Theorem 14–2).
7. Show that $(7, 9)$, $(10, -3)$, and $(2, -5)$ are the coordinates of the vertices of a right triangle.
8. Show that $(1, 4)$, $(3, 5)$, $(-3, 12)$, and $(-1, 13)$ are the coordinates of the vertices of a rectangle.
9. Show that $(-3, -3)$, $(-1, -2)$, $(1, -6)$, and $(-1, -7)$ are the coordinates of the vertices of a rectangle.
10. Show that $(10, 2)$, $(8, 8)$, $(-1, 5)$, and $(1, -1)$ are the coordinates of the vertices of a rectangle.

Activity

Suppose $\overline{AB} \parallel \overline{CD}$, $\overline{EF} \perp \overline{AB}$, and $\overleftrightarrow{EF} \perp \overleftrightarrow{CD}$ with coordinates as shown.



a. Using tracing paper or a MIRA, reflect $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ about the x-axis. Do the same relationships hold true with the reflected segments?

b. Reflect the segments about the y-axis. Do the same relationships hold true for these reflected segments?

c. Reflect the segments about the line $y = x$. Do the same relationships hold true for these reflected segments?

11. The vertices of a triangle have coordinates $(5, 1)$, $(-1, 7)$, and $(1, -3)$. Find the slopes of the three sides. Find the slopes of the three altitudes.

B.

12. Find y so that the line through $(-4, -3)$ and $(8, y)$ will be parallel to the line through $(4, -4)$ and $(3, 5)$.

13. Find y so that the line through $(-2, -1)$ and $(10, y)$ will be perpendicular to the line through $(6, -2)$ and $(5, 7)$.

14. Determine a so that the line through $(7, 1)$ and $(4, 8)$ will be parallel to the line through $(2, a)$ and $(a, -2)$.

15. Determine b so that the line through $(2, 3)$ and $(4, -5)$ is perpendicular to the line through $(4, -5)$ and (b, b).

16. The coordinates of A, B, and C are $(-3, 2)$, $(4, -2)$ and $(0, 6)$, respectively. Find D such that $\overline{AB} \parallel \overline{CD}$ and D is on the x-axis.

17. If the coordinates of A and B are $(0, 4)$ and $(-5, 1)$, and $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$, find the point at which $\overleftrightarrow{AC}$ crosses the x-axis.

C.

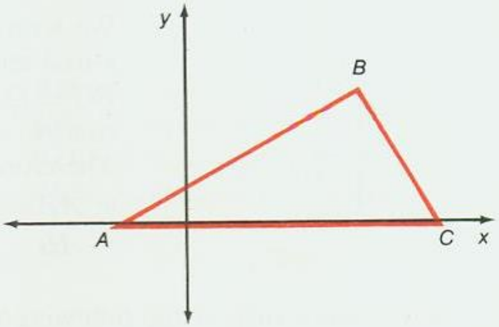
18. $ABCD$ is a rhombus with vertices $A(-3, 6)$, $B(5, 7)$, and $C(9, 0)$. Find the coordinates of D.

19. $ABCD$ is a parallelogram with vertices $A(3, 6)$, $B(5, 9)$, and $C(8, 2)$. Find the coordinates of D.

20. A circle with radius a and center at the origin contains the point with coordinates (c, d). Find the slope of the tangent to the circle at the point (c, d).

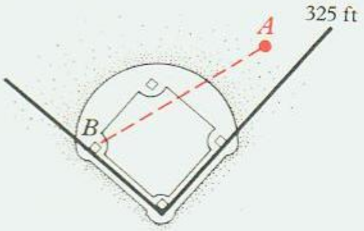
PROBLEM SOLVING

Consider the points $A(-2, 0)$, $B(6, 4)$, and $C(x, 0)$. If $\overline{AB} \perp \overline{BC}$, find the area of $\triangle ABC$. (*Hint:* First find the coordinates of point C.)



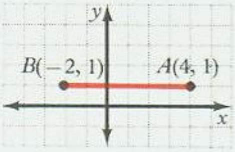
14-5 The Distance Formula

In this section we will study one of the most important formulas in analytic geometry—the *Distance Formula*. We can use the Distance Formula to find the distance the ball was thrown (see the picture on the right).



If a right fielder throws the ball from point A to third base (point B), how far did the ball travel?

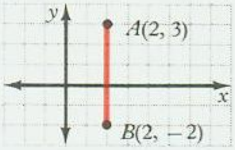
Example 1 To find the length of a segment parallel to the x-axis.



$AB = |4 - (-2)|$
$AB = |6|$
$AB = 6$

If $\overline{AB}$ is parallel to the x-axis, and the coordinates of A and B are (x_1, y_1) and (x_2, y_1), then $AB = |x_1 - x_2|$.

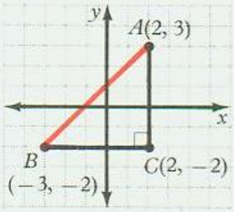
Example 2 To find the length of a segment parallel to the y-axis.



$AB = |3 - (-2)|$
$AB = |5|$
$AB = 5$

If $\overline{AB}$ is parallel to the y-axis, and the coordinates of A and B are (x_1, y_1) and (x_1, y_2), then $AB = |y_1 - y_2|$.

Example 3 To find the length of a segment not parallel to either axis.



We wish to find the length of $\overline{AB}$. Let $\overline{BC}$ be parallel to the x-axis and $\overline{AC}$ be parallel to the y-axis. Then $BC = |2 - (-3)|$ and $AC = |3 - (-2)|$. $\triangle ABC$ is a right triangle. So by the Pythagorean Theorem, $AB^2 = BC^2 + AC^2$. Therefore, $AB^2 = (2 - (-3))^2 + (3 - (-2))^2$

or $AB = \sqrt{(2 + 3)^2 + (3 + 2)^2}$

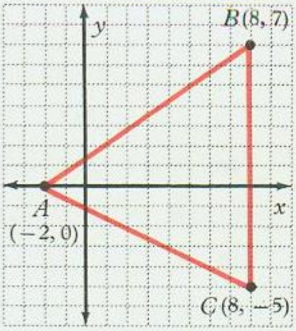
$AB = \sqrt{25 + 25} = 5\sqrt{2}$

These examples suggest the following theorem.

Theorem 14–4 **The Distance Formula.** If A has coordinates (x_1, y_1) and B has coordinates (x_2, y_2) then $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example

Use the Distance Formula to decide whether $\triangle ABC$ is isosceles.



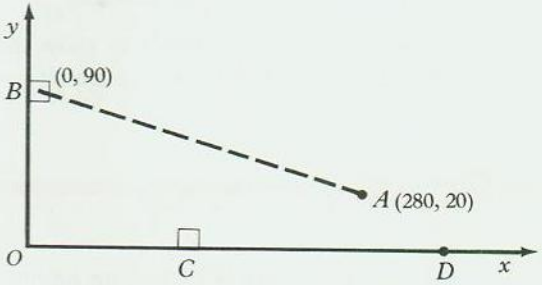
We begin by finding the lengths of the three sides.

$$AB = \sqrt{(8-(-2))^2 + (7-0)^2} = \sqrt{149}$$
$$BC = \sqrt{0^2 + (7-(-5))^2} = \sqrt{144}$$
$$AC = \sqrt{(8-(-2))^2 + (-5-0)^2} = \sqrt{125}$$

No two sides are the same length, so $\triangle ABC$ is not isosceles.

APPLICATION

Now we return to the question posed at the beginning of this section: "How far did the fielder throw the ball?" Assign a coordinate system as shown on the right with home plate as the origin. The player will be assigned a position of (280, 20). Using the Distance Formula, we obtain:



B (0, 90)—third base
C (90, 0)—first base
D (325, 0)—right field fence
A (280, 20)—position of player

$$AB = \sqrt{(280-0)^2 + (20-90)^2}$$
$$AB = \sqrt{(280)^2 + (-70)^2}$$
$$AB = \sqrt{78{,}400 + 4900}$$
$$AB = \sqrt{83{,}300}$$
$$AB \doteq 288.62 \text{ feet}$$

The ball traveled about 289 feet.

EXERCISES

A.

In exercises 1-8, find the distances between the given points.

1. $(-4, 5)$ and $(6, 5)$

2. $(3, 2)$ and $(3, -8)$

3. $(4, 5)$ and $(-3, -2)$

4. $(-2, 5)$ and $(5, -2)$

5. $(4, 0)$ and $(0, -6)$

6. $(1, 2)$ and $(-7, 3)$

7. $(-5, 3)$ and $(0, -4)$

8. $(6, -2)$ and $(7, 3)$

In exercises 9-12, use the Distance Formula to classify the triangle as scalene, isosceles, or equilateral.

9. $A(4, 5)$, $B(5, -2)$, and $C(1, 1)$.

10. $A(1, 1)$, $B(-3, 5)$, and $C(2\sqrt{3} - 1, 2\sqrt{3} + 3)$

11. $A(-6, 2)$, $B(1, 7)$, and $C(6, 3)$.

12. $A(10, -5)$, $B(-2, 1)$, and $C(7, 4)$.

In exercises 13 and 14, use the Distance Formula to determine whether or not the triangle is a right triangle.

13. $A(1, 4)$, $B(-2, -2)$, and $C(10, -8)$

14. $A(5, 7)$, $B(8, -5)$, and $C(-4, -4)$

15. Is it easier to use Definition 14-1 or Theorem 14-4 to show that a triangle is a right triangle?

16. Use the Distance Formula to show that $ABCD$ is a parallelogram if the vertices are $A(1, -5)$, $B(7, -1)$, $C(2, 0)$, and $D(-4, -4)$.

Activity

There is another way to determine points in a plane besides using Cartesian coordinates. For example, to locate P you could say, "Travel at a 45° angle for 4 units."

The notation for this is represented by the ordered pair (4, 45°) or in general (r, θ) where θ represents the angle and r represents the distance from O to the given point.

Draw a ray $\overrightarrow{OM}$ (with O the origin) and then use your protractor to graph the following points:

1. (2, 30°)

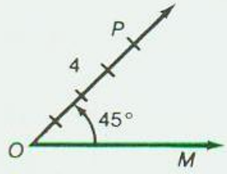
2. (3, 90°)

3. (5, 90°)

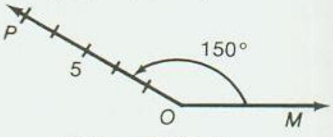
4. (2.5, 120°)

5. (4, 42°)

6. (1, 180°)



$(r, \theta) = (4, 45°)$



$(r, \theta) = (5, 150°)$

17. Find the length of the median $\overline{AD}$ in $\triangle ABC$ with vertices $A(2, 6)$, $B(3, -5)$, and $C(-1, 7)$.

In exercises 18–20, use the Distance Formula to determine if B is between A and C. (If B is between A and C, then $AB + BC = AC$.)

18. $A(-3, -7)$, $B(0, -2)$, $C(6, 8)$

19. $A(1, -2)$, $B(4, 3)$, $C(10, 12)$

20. $A(1, 4)$, $B(2, 3)$, $C(4, 1)$

B.

21. Find x if the distance between $(1, 2)$ and $(x, 8)$ is 10.

22. $ABCD$ is a rectangle inscribed in a circle with vertices $A(0, 0)$, $B(2, 1)$, $C(4, -3)$, and $D(2, -4)$. Find the length of the diameter of the circle.

23. Find the area of $\triangle ABC$ with vertices $A(-3, -4)$, $B(3, 4)$, and $C(-5, 0)$.

24. Find the area of $ABCD$ with vertices $A(-2, 3)$, $B(3, 8)$, $C(8, 3)$ and $D(3, -2)$.

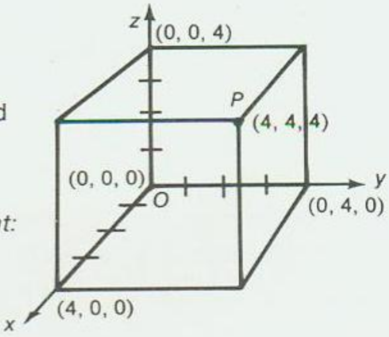
C.

25. Find the coordinates of the point equidistant from $(3, 11)$, $(9, 5)$, and $(7, -1)$.

26. Find the coordinates of the point that is equidistant from $(0, 6)$ and $(10, 0)$ and lies on the line $y = x$.

27. Graph at least four points that satisfy the condition that the distance from the point $(1, 2)$ is always 5 units.

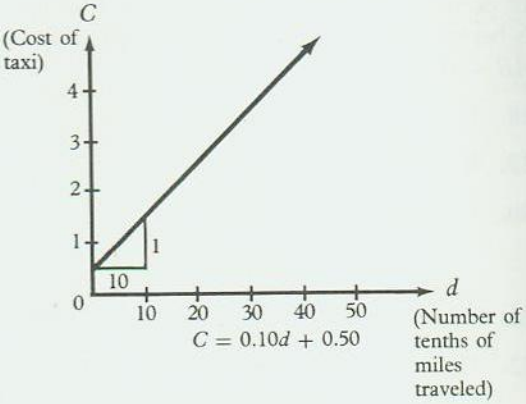
PROBLEM SOLVING



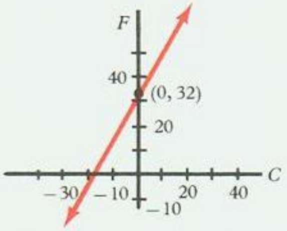
Consider a three-dimensional coordinate system with an x-axis, a y-axis, and a z-axis. A cube with each edge of 4 units is placed as shown in the diagram.

1. Find the length of $\overline{OP}$.
2. Find a formula for the length of $\overline{OP}$ in a cube of any size. (*Hint:* Let P have coordinates (x_1, y_1, z_1).)

14–6 The Equation of a Line



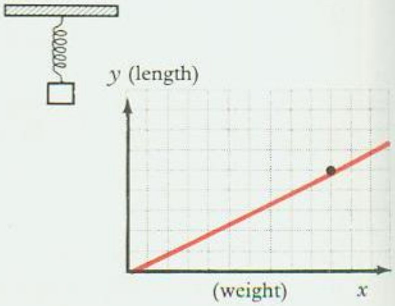
Relationships in our world can often be represented by a straight line graph. These relationships, such as the taxi fares shown above, are called *linear relationships.* Every linear relationship can be represented by an equation of the form $y = mx + b$ where the line cuts the y-axis at b and has slope m.



Example 1 The relationship between the Fahrenheit and the Celsius scales is given by the equation

$$F = \tfrac{9}{5}C + 32.$$

The slope is $\frac{9}{5}$. The F-intercept is 32.



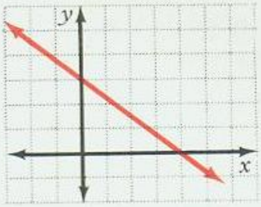
Example 2 Suppose for every ounce of weight added to a spring it stretches an additional 0.5 inches. The relationship between the weight (x) and the length of the stretch (y) is given by the equation

$$y = 0.5x.$$

The slope is 0.5. The y-intercept is 0.

Theorem 14–5 Any straight line in the coordinate plane that is not parallel to the y-axis can be represented by the equation $y = mx + b$, where m is the slope and b is the y-intercept.

Example 1 Determine the slope and y-intercept of a line given an equation of the form $ax + by = c$.



$3x + 4y = 12$

Consider the equation $3x + 4y = 12$.
Solve for y. $3x + 4y = 12$

$$4y = -3x + 12$$
$$y = -\tfrac{3}{4}x + 3$$

Now read the slope and y-intercept. By Theorem 14-5

$$m = -\tfrac{3}{4} \quad \text{(the slope) and}$$
$$b = 3 \quad (y\text{-intercept}).$$

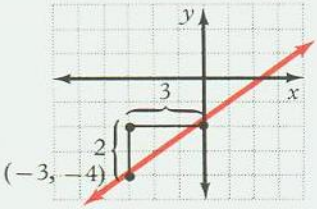
The following two examples illustrate how Theorem 14-5 can be used to find the equation of a line given certain conditions.

Example 2 Find the equation of a line given the slope of the line and a point on the line.

Consider a line with slope $\frac{2}{3}$ and a point $(-3, -4)$.

Step 1 The slope is determined by the equation $m = \dfrac{y_1 - y_2}{x_1 - x_2}$.
Consider the two points (x, y) and $(-3, -4)$ and obtain the equation

$$\frac{2}{3} = \frac{y - (-4)}{x - (-3)}.$$



$y = \frac{2}{3}x - 2$

Step 2 Simplify the equation.

$$2(x - (-3)) = 3(y - (-4))$$
$$\text{or } y = \tfrac{2}{3}x - 2$$

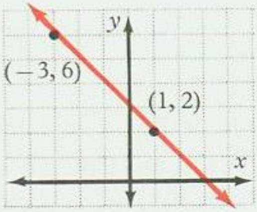
Example 3 Find the equation of a line given two points on the line.

Consider a line which contains the points $(-3, 6)$ and $(1, 2)$.

Step 1 The slope $m = \dfrac{6 - 2}{-3 - 1} = \dfrac{4}{-4} = -1$

Step 2 Since the slope and a point are known, follow Example 2 using the points (x, y) and $(-3, 6)$.

$$-1 = \frac{y - 6}{x - (-3)} \text{ or } y = -x + 3$$



$y = -x + 3$

EXERCISES

A.

In exercises 1–6, write the equation of a line in the form $y = mx + b$.

1. $m = 2, b = 3$

2. $m = \frac{2}{3}, b = -2$

3. $m = -\frac{3}{4}, b = 5$

4. $m = -0.2, b = -2.5$

5. $m = 10, b = -4$

6. $m = 0, b = 2$

In exercises 7–10, graph the equation.

7. $y = x + 4$

8. $y = -2x - 3$

9. $y = \frac{3}{4}x - 5$

10. $y = -\frac{3}{2}x + 3$

In exercises 11–16, find the slope and *y*-intercept of each line and then graph the equation.

11. $y + 2x = 4$

12. $2y - x = 5$

13. $3x - 2y = 4$

14. $\frac{1}{2}y + \frac{1}{3}x = 1$

15. $5 - 3x = 2y$

16. $3x - 4y = 12$

In exercises 17–22, find the equation of the line given the slope and a point.

17. $m = 2, (1, -3)$

18. $m = -3, (-2, 1)$

19. $m = \frac{2}{3}, (0, 0)$

20. $m = -\frac{1}{2}, (4, 3)$

21. $m = \frac{3}{2}, (-3, -4)$

22. $m = -\frac{5}{2}, (0, 4)$

In exercises 23–28, find the equation of the line that contains the given points.

23. $(0, 0), (4, 3)$

24. $(-2, 1), (5, -3)$

25. $(0, 0), (-3, -3)$

26. $(0, -4), (6, 0)$

27. $(5, 2), (-3, 2)$

28. $(1, 3), (-4, -2)$

Activity

Identify two variables that you think have some relationship. For example:

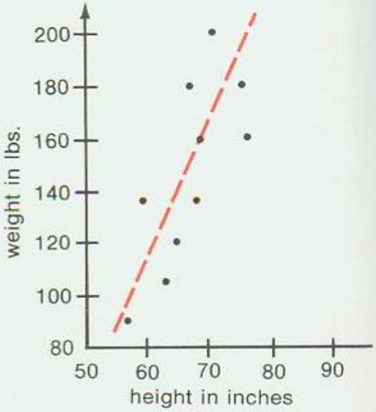
a. a person's height and weight

b. the number of "at bats" and the number of "hits" for a baseball player

c. the circumference of various circles and corresponding diameters.

Then,

1. Graph the ordered pairs.
2. Draw a line that is the closest fit to the data.
3. Determine the slope and the intercept of the line.



B.

In exercises 29–31, find the slope of a line that is perpendicular to the given line.

29. $y = -2x + 3$

30. $2x - 3y = 6$

31. $12x + 30y = 18$

32. The vertices of a triangle have coordinates (0, 0), (2, 4), and (−4, 2). Find the equations of the sides of the triangle.

33. Find the point of intersection of the line $x - 3y = 1$ and the line containing the points (1, 7) and (6, −3).

34. If the x- and y-intercepts of a line are (4, 0) and (0, −3), what is the equation of the line?

35. Find the area of a triangle formed by the x-axis, the y-axis, and the line $y = x - 5$.

36. Find the equation of the median $\overline{AD}$ of $\triangle ABC$ with vertices $A(4, 4)$, $B(6, 2)$, and $C(-2, -4)$.

37. Find the equations of the diagonals of rectangle $ABCD$ with vertices $A(-6, -4)$, $B(-6, 2)$, $C(3, 2)$, and $D(3, -4)$.

38. Find the equation of the altitude $\overline{AD}$ of $\triangle ABC$ with vertices $A(-1, 5)$, $B(-7, -3)$, and $C(5, 1)$.

39. Find the equation of the line that is the perpendicular bisector of the segment $\overline{AB}$ with endpoints $A(10, 2)$ and $B(2, -6)$.

40. Find the equation of the line that contains the point (4, 2) and is perpendicular to the line $y = -2x - 4$.

C.

41. A diagonal of a square lies on the line $3x - 5y = 14$. One vertex is at (0, 4). Find the equation of the other diagonal.

42. The equations of two adjacent sides of a parallelogram are $x + 2y - 4 = 0$ and $3x + y + 3 = 0$. One vertex has coordinates (8, −7). Find the equations of the other two sides.

43. Find the distance between the parallel lines with equations $y = -1x + 10$ and $y = -1x + 15$. (*Hint:* Draw an accurate diagram.)

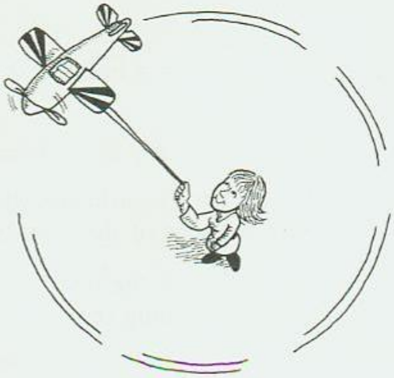
44. The coordinates of $\triangle ABC$ are $A(0, 0)$, $B(6, 0)$, and $C(4, 6)$. $\overline{AD}$ is an altitude to $\overline{BC}$. Find the coordinates of D.

45. The coordinates of the vertices of a triangle are (0, 0), (18, 0), and (6, 12). Find the coordinates of the centroid (the point of intersection of the medians).

PROBLEM SOLVING

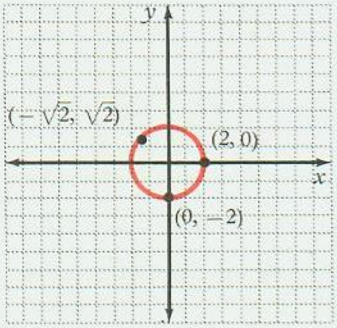
Find the distance between the point (2, 1) and the line $3x - 4y = 0$. (*Hint:* Draw an accurate figure and label the distance you are asked to find.)

14–7 The Equation of a Circle

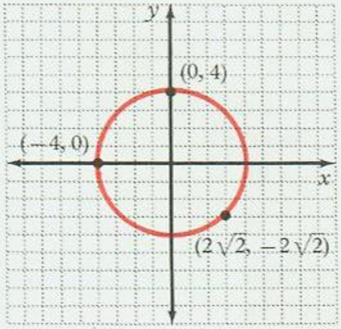


Suppose a coordinate system is assigned so that the girl is at the origin. How could the path of the model airplane be described mathematically? The theorem in this lesson will help to answer this question.

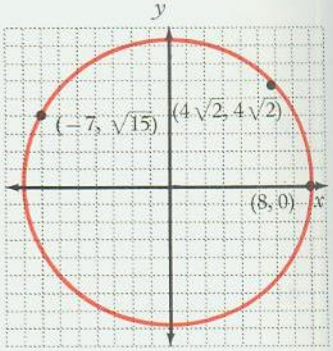
In the examples below, show that the points are on the circle. This can be done by showing that the coordinates of the points satisfy the equation given for the circle.



$x^2 + y^2 = 4$



$x^2 + y^2 = 16$



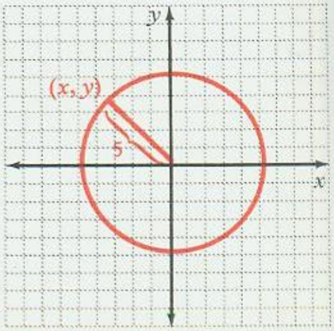
$x^2 + y^2 = 64$

Consider the circle on the right with its center at the origin and a radius of 5 units. Suppose (x, y) is a point on the circle. The Distance Formula tells us that

$$\sqrt{(x-0)^2 + (y-0)^2} = 5.$$

Hence $x^2 + y^2 = 25$, which is the equation of the circle.

This suggests the following theorem.

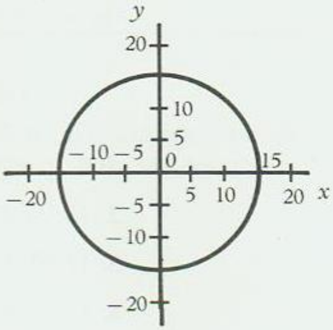


Theorem 14–6 The graph of the equation $x^2 + y^2 = r^2$ is a circle with radius r and center at the origin.

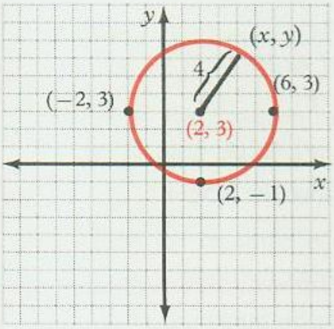
APPLICATION

Consider the girl flying the model airplane as described at the beginning of this section. If the girl is at the origin and if the string is 15 feet long, then by Theorem 14–6, the path of the plane is described by the equation

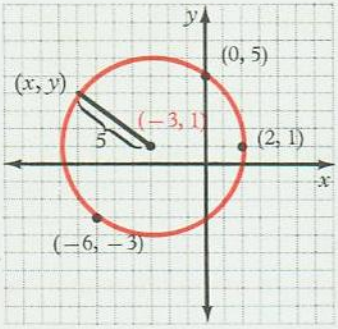
$$x^2 + y^2 = 225.$$



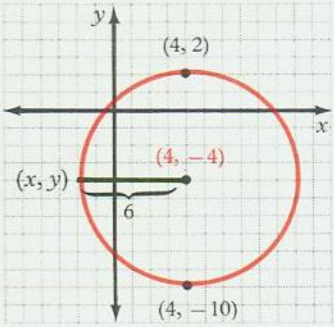
In the examples below, show that the points are on the circles. The coordinates of the center of the circle are given in red.



$(x - 2)^2 + (y - 3)^2 = 16$



$(x - (-3))^2 + (y - 1)^2 = 25$
or $(x + 3)^2 + (y - 1)^2 = 25$



$(x - 4)^2 + (y - (-4))^2 = 36$
or $(x - 4)^2 + (y + 4)^2 = 36$

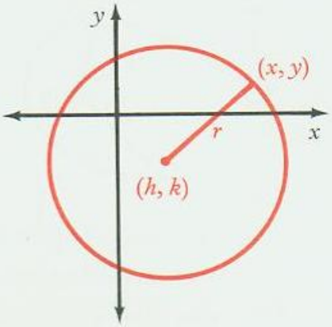
Suppose the circle on the right has a center at the point (h, k) and has a radius of r units. The Distance Formula tells us that

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

Hence, the equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2.$$

We now state the following theorem.



Theorem 14–7 The graph of the equation $(x - h)^2 + (y - k)^2 = r^2$ is a circle with radius r and center at the point (h, k).

EXERCISES

A.

In exercises 1–9, write the equation of a circle with the given center and radius. Then graph the circle using your compass.

1. $(0, 0)$; 5
2. $(0, 0)$; 2
3. $(0, 0)$; 6
4. $(0, 0)$; 4
5. $(2, 3)$; 6
6. $(-3, -4)$; 4
7. $(5, 2)$; $\sqrt{2}$
8. $(-4, 6)$; 2.5
9. $(0, -4)$; 5

In exercises 10–17, find the center and the radius of the circle.

10. $x^2 + y^2 = 25$
11. $x^2 + y^2 = 36$
12. $x^2 + y^2 = 20$
13. $(x - (-3))^2 + (y - 4)^2 = 25$
14. $(x + 4)^2 + (y - 2)^2 = 10$
15. $x^2 + (y - 3)^2 = 12$
16. $(x + 2)^2 + y^2 = 9$
17. $(x + 2)^2 + (y + 4)^2 = 36$

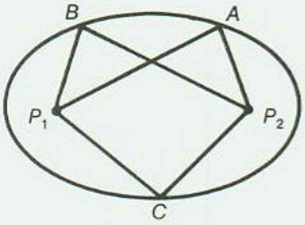
B.

18. Write the equation of a circle with center at $(-4, 0)$ and containing the origin.

19. Write the equation of a circle with center at $(3, 4)$ and containing the origin.

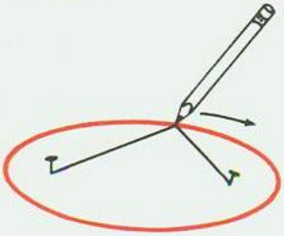
Activity

An ellipse is a figure closely related to a circle. Study the definition below.



P_1 and P_2 are fixed points. The sum of the distances from any point on the curve to P_1 and P_2 is always the same. That is, $P_1B + BP_2 = P_1A + AP_2 = P_1C + CP_2$.

An *ellipse* is the set of points in which the sum of the distances from two fixed points (P_1 and P_2) to a point on the curve is a constant.



Here is how you can draw an ellipse.

1. Fix two points with tacks.
2. Tie one end of a string to one tack and the other end to the second tack.
3. Move the pencil as suggested in the picture.

Draw several ellipses by changing the distance between the tacks and by changing the length of the string. When does the ellipse look more like a circle?

20. Write the equation of a circle with center at $(-2, 3)$ and containing the point $(3, 3)$.

21. Write the equation of a circle with a diameter whose endpoints are $(-4, 0)$ and $(2, 0)$.

22. Write the equation of a circle with a diameter whose endpoints are $(3, 6)$ and $(3, -2)$.

23. Write the equation of a circle with a diameter whose endpoints are $(-4, 8)$ and $(6, 2)$.

24. Write the equation of a line that is tangent to the circle $x^2 + y^2 = 25$ at the point $(-3, 4)$. (Remember that the tangent is perpendicular to the radius.)

25. Write the equation of the line of centers for the circles $(x + 4)^2 + (y - 2)^2 = 36$ and $(x - 5)^2 + (y + 3)^2 = 17$.

26. Write the equation of the circle with center at $(-5, -5)$ and tangent to both the x- and y-axis.

27. Write the equation of a circle that has the same center as the circle $(x + 4)^2 + (y - 3)^2 = 9$ and is tangent to the y-axis.

C.

28. Find the equation of the circle whose center is on the line $y = \frac{1}{2}x$ and that contains the points $(0, 6)$ and $(0, -2)$.

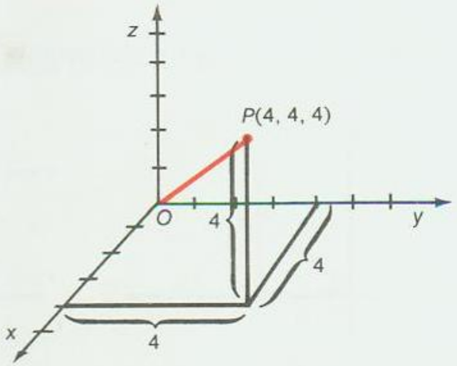
29. Write the equation of the circle with center at $(1, 7)$ and tangent to the line $x + 3y = 12$.

30. Find the length of a tangent from $(6, 4)$ to the circle $x^2 + y^2 = 36$.

PROBLEM SOLVING

Suppose a sphere with center at (0, 0, 0) contains the point $P(4, 4, 4)$ in a three-dimensional coordinate system. Write the equation that represents the sphere.

Write the equation of the sphere with center at (1, 3, 2) and containing the point (4, −2, 3).



14–8 Using Coordinates to Prove Theorems

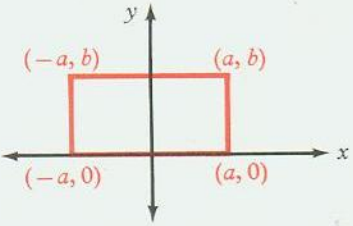
Some theorems can be easily proved using coordinates; however, it is very important to choose the coordinates carefully. Study the following two examples.

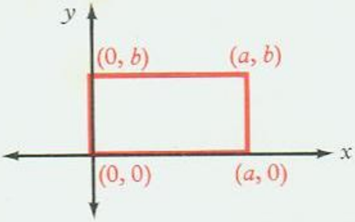
Example 1

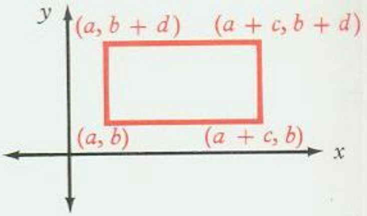
Consider the theorem:

The diagonals of a rectangle are congruent.

Which of the following placements of the rectangles seems the best?







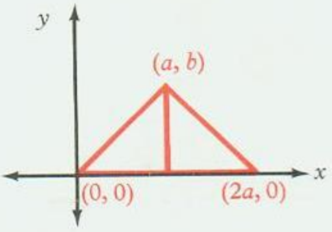
The theorem could be proved using the Distance Formula with any of the three positions. However, the middle figure may be the easiest to use since the coordinates needed have more zeroes.

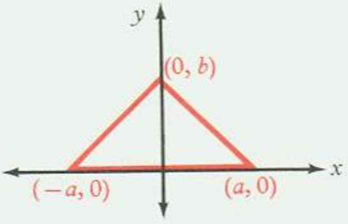
Example 2

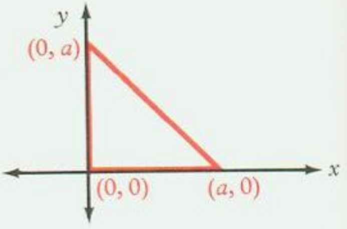
Consider this theorem:

The median to the base of an isosceles right triangle is perpendicular to the base.

Which of the following placements of an isosceles right triangle seems best to you?

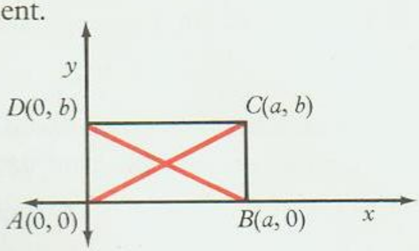






Each of the three positions could be used to prove the theorem. However, we will use the position on the right in the proof on the next page.

Example 3 Theorem: The diagonals of a rectangle are congruent.



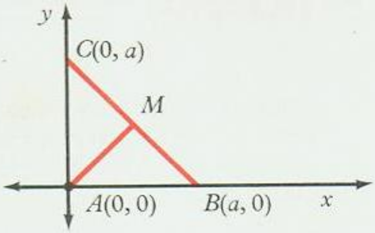
PROOF

Given: $ABCD$ is a rectangle with coordinates $A(0, 0)$, $B(a, 0)$, $C(a, b)$, $D(0, b)$ and diagonals $\overline{AC}$ and $\overline{BD}$.

Prove: $AC = BD$

Statements	Reasons
1. $ABCD$ is a rectangle with $A(0, 0)$, $B(a, 0)$, $C(a, b)$, $D(0, b)$.	1. Given
2. $AC = \sqrt{(a - 0)^2 + (b - 0)^2}$	2. Distance Formula
3. $AC = \sqrt{a^2 + b^2}$	3. Properties of numbers
4. $BD = \sqrt{(a - 0)^2 + (0 - b)^2}$	4. Why?
5. $BD = \sqrt{a^2 + b^2}$	5. Properties of numbers
6. $AC = BD$	6. Why?

Example 4 Theorem: The median to the base of an isosceles right triangle is perpendicular to the base.



PROOF

Given: Isosceles right triangle ABC with $A(0, 0)$, $B(a, 0)$, $C(0, a)$ and $BM = MC$.

Prove: $\overline{AM} \perp \overline{BC}$

Statements	Reasons
1. Isosceles right triangle ABC with $A(0, 0)$, $B(a, 0)$, $C(0, a)$	1. Given
2. The coordinates of M are $\left(\frac{a}{2}, \frac{a}{2}\right)$.	2. Midpoint Formula (Theorem 14–1)
3. Slope of $\overline{AM} = \dfrac{\frac{a}{2} - 0}{\frac{a}{2} - 0} = 1$	3. Definition of slope
4. Slope of $\overline{BC} = \dfrac{a - 0}{0 - a} = -1$	4. Definition of slope
5. Slope of $\overline{AM}$ · slope of $\overline{BC} = (1)(-1) = -1$	5. Properties of numbers
6. $\overline{AM} \perp \overline{BC}$	6. Why?

EXERCISES

A.

In exercises 1–5, draw a coordinate system and position the figure. Give a statement about the figure that could be proved.

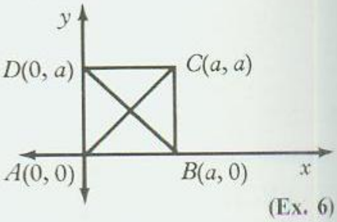
1. square

2. equilateral triangle

3. parallelogram

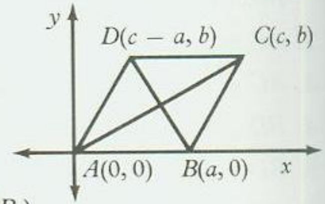
4. isosceles trapezoid

5. a trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $m\angle A = m\angle D = 90$



(Ex. 6)

6. Given: $ABCD$ is a square with $A(0,0)$, $B(a,0)$, $C(a,a)$, and $D(0,a)$.
Prove: $AC = BD$



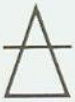
7. Given: $ABCD$ is a rhombus with $A(0,0)$, $B(a,0)$, $C(c,b)$, and $D(c-a,b)$.
Prove: $\overline{AC} \perp \overline{BD}$
(*Hint:* Show slope of $\overline{AC} \cdot$ slope of $\overline{BD} = -1$. Remember $AD = AB$.)

Activity

Obtain styrofoam cones (or cones made of some other material). Slice the cones in various ways as suggested below. Note the type of figure formed.

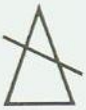
slice **figure**

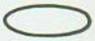
1. Slice a section parallel to the base. The figure formed is a *circle*.



circle

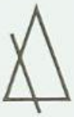
2. Slice a section that is not parallel to the base and does not intersect the base. The figure formed is called an *ellipse*. (See previous activity.)

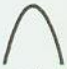




ellipse

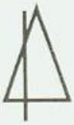
3. Slice a section parallel to the slant edge of the cone. The figure formed is called a *parabola*.

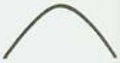




parabola

4. Slice a section perpendicular to the base. The figure formed is called a *hyperbola*.

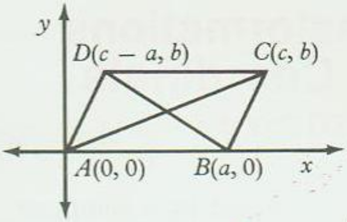




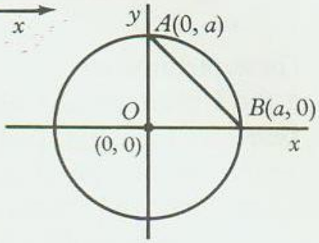
hyperbola

Can you see why these figures are called conics?

8. Given: *ABCD* is a parallelogram with $A(0, 0)$, $B(a, 0)$, $C(c, b)$, and $D(c - a, b)$.
Prove: $\overline{AC}$ and $\overline{BD}$ bisect each other.



9. Given: Circle with center at the origin and chord $\overline{AB}$ with coordinates $(0, a)$ and $(a, 0)$
Prove: The perpendicular bisector of a chord of a circle contains the center.



B.

10. Prove: The diagonals of a square are perpendicular. (Assign your coordinate system as shown in exercise 6.)

11. Prove: The diagonals of an isosceles trapezoid *ABCD* are congruent. (Assign the vertices $A(0, 0)$, $B(a, 0)$, $C(c, b)$, and $D(a - c, b)$.)

12. Prove: The midpoint of the hypotenuse of a right triangle *ABC* is equidistant from the vertices. (Assign the vertices $A(0, 0)$, $B(a, 0)$, and $C(0, b)$.)

13. Prove: If the diagonals of parallelogram *ABCD* are congruent, the parallelogram is a rectangle. (Assign the vertices $A(0, 0)$, $B(a, 0)$, $C(c, b)$ and $D(c - a, b)$.)

C.

14. Prove: If the diagonals of a trapezoid *ABCD* are congruent, then the legs of the trapezoid are congruent.

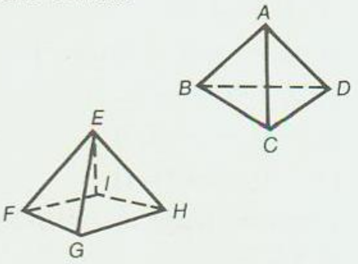
15. Prove: The line segment joining the midpoints of two sides of a triangle is equal to one half of the third side and is parallel to it.

PROBLEM SOLVING

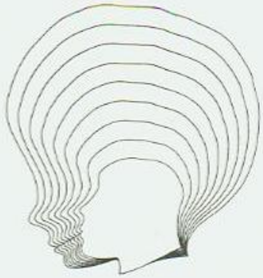
The following problem appeared on a college entrance examination. The intended answer was 7 faces. However, a high school student argued that 7 faces was not the correct answer. Was he right? If so, what is the correct answer?

In pyramids *ABCD* and *EFGHI* all faces except base *FGHI* are congruent equilateral triangles. If face *ABC* were placed on face *EFG* so that the vertices of the triangle coincide, how many exposed faces would the resulting solid have?

a. 5 b. 6 c. 7 d. 8 e. 9



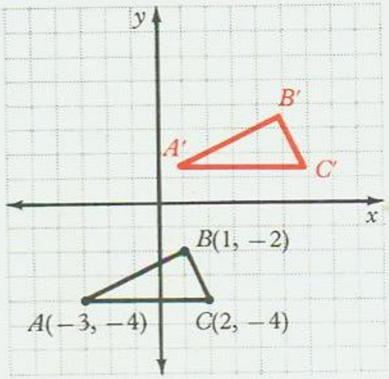
14–9 Transformations and Coordinate Geometry



These pictures were generated by a computer at the University of Connecticut (see *Scientific American,* February 1980). The computer used a special type of transformation to show the sequence (top picture) that proceeds from infancy (inside profile) to adulthood (outside profile). A different type of transformation was used to produce the sequence (bottom picture) that proceeds from a "Neanderthal Man" (inside profile) to a futuristic being (outside profile). Transformations help simulate growth and changes in the body characteristics of people so they may be studied by scientists.

The effects of simple transformations, such as translations, reflections, rotations, and others, can be shown on a coordinate axis. Study this example.

Example: To graph a figure and its image after a transformation, when the transformation rule is $(x, y) \rightarrow (x + 4, y + 5)$.



Step 1 Graph a figure, for example $\triangle ABC$.
$A(-3, -4)$ $B(1, -2)$ $C(2, -4)$

Step 2 Apply the transformation rule to points A, B, and C to get image points A', B', and C'.

$$(x, y) \longrightarrow (x + 4, y + 5)$$
$$(-3, -4) \longrightarrow (1, 1)$$
$$(1, -2) \longrightarrow (5, 3)$$
$$(2, -4) \longrightarrow (6, 1)$$

Step 3 Graph the image of $\triangle ABC$, that is, $\triangle A'B'C'$.

This transformation is a translation.

EXERCISES

A.

Graph each figure and its image using the transformation rule given in exercises 1–3.

1. Figure: Triangle ABC, $A(-6, 3)$, $B(-4, 5)$, $C(-3, 4)$
   Transformation Rule: $(x, y) \rightarrow (x + 4, y + 3)$

2. Figure: Triangle DEF, $D(2, 1)$, $E(3, 4)$, $F(1, 5)$
   Transformation Rule: $(x, y) \rightarrow (-x, y)$

3. Figure: Triangle PQR, $P(7, 1)$, $Q(7, 4)$, $R(5, 2)$
   Transformation Rule: $(x, y) \rightarrow (-y, x)$

4. For exercises 1–3, tell whether the transformation is a translation, a rotation, or a reflection. If it is a translation, give the distance the figure is translated. If it is a reflection, specify the line of reflection. If it is a rotation, specify the center and the number of degrees.

B.

For exercises 5–7, graph the quadrilateral and its image. Then answer the questions.

5. Quadrilateral: $A(1, 1)$, $B(1, 2)$, $C(2, 1)$, $D(2, 2)$.
   Transformation Rule: $(x, y) \rightarrow (3x, 3y)$
   This transformation is called a *magnification*. How do the lengths of the sides of the figure compare with the lengths of the sides of the image? How do the areas compare?

6. Quadrilateral: $P(1, 2)$, $Q(1, 3)$, $R(3, 3)$, $S(3, 2)$
   Transformation Rule: $(x, y) \rightarrow (x + 3y, y)$
   This transformation is called a *shear*. How does the area of the image compare with the area of the quadrilateral?

7. Quadrilateral: $W(4, -1)$, $X(3, 2)$, $Y(-3, 2)$, $Z(-2, 1)$
   Transformation Rule: $(x, y) \rightarrow (\frac{1}{2}x, 2y)$
   This transformation is called a *stretch*. How does it change a figure? How do the areas compare?

C.

8. Make up some transformation rules and graph the figure and its image to see how the figure is transformed. Can you make a rule that "shrinks" a figure?

Important Ideas—Chapter 14

Terms

Cartesian coordinate system (p. 505)
Slope of a line (p. 512)

Theorems

14-1 If the coordinates of the segment $\overline{P_1P_2}$ are (x_1, y_1) and (x_2, y_2), then the coordinates of the midpoint of $\overline{P_1P_2}$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

14-2 The product of the slopes of two perpendicular lines is -1. (Assuming neither line is parallel to the y-axis.)

14-3 The slopes of two parallel lines are equal. (Assuming neither line is parallel to the y-axis.)

14-4 **The Distance Formula.** If A has coordinates (x_1, y_1) and B has coordinates (x_2, y_2), then $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

14-5 Any straight line in the coordinate plane that is not parallel to the y-axis can be represented by the equation $y = mx + b$, where m is the slope and b is the y-intercept.

14-6 The graph of the equation $x^2 + y^2 = r^2$ is a circle with radius r and center at the origin.

14-7 The graph of the equation $(x - h)^2 + (y - k)^2 = r^2$ is a circle with radius r and center at the point (h, k).

Chapter 14—Review

1. Find the distances between the coordinates.
   a. $(1, 2)$ and $(5, 8)$ **b.** $(-3, 4)$ and $(5, -2)$
   c. $(-4, -2)$ and $(2, -2)$

2. Find the slopes of the lines containing the coordinates in exercise 1.

3. Find the midpoints of the segments determined by the coordinates in exercise 1.

4. Find the equation of a line that contains the point $(2, -4)$ with a slope of -3.

5. Find the equation of a line that contains the point $(4, 1)$ and is parallel to the line $3x + 4y = 2$.

6. Show that $(8, -5)$, $(0, -7)$, and $(5, 7)$ are the vertices of a right triangle.

7. Find the missing coordinate so that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ for $A(2, 5)$, $B(4, -2)$, $C(-3, -4)$ and $D(6, x)$.

8. Find the missing coordinate so that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ for $A(-4, -5)$, $B(7, -2)$, $C(-4, 6)$ and $D(x, 0)$.

9. Give the center and radius of the circle with equation $(x - 2)^2 + (y + 3)^2 = 24$.

10. Write the equation of a circle with center at the origin which passes through the point $(4, 3)$.

11. **Given:** $ABCD$ is a square with vertices $A(0, 0)$, $B(a, 0)$, $C(a, a)$, and $D(0, a)$.
    Prove: $\overline{AC} \perp \overline{BD}$

12. **Given:** Figure $WXYZ$ with vertices $W(0, 0)$, $X(a, 0)$, $Y(a + b, c)$, and $Z(b, c)$
    Prove: $WXYZ$ is a parallelogram.

Chapter 14—Test

1. Find the distances between the coordinates.

   a. $(3, 6)$ and $(-6, -3)$ **b.** $(-4, 0)$ and $(0, -4)$ **c.** $(-2, 5)$ and $(-2, -1)$

2. Find the slopes of the lines containing the coordinates in exercise 1.

3. Find the midpoints of the segments determined by the coordinates in exercise 1.

4. Find the equation of a line that contains the points $(-3, 5)$ and $(2, -4)$.

5. Find the equation of a line that contains the point $(3, 5)$ and is perpendicular to the line $2x + 6y = -3$.

6. Show that $(4, 5)$, $(6, 4)$, $(3, -1)$, and $(1, 0)$ are the coordinates of the vertices of a parallelogram.

7. Find the missing coordinate so that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ for $A(-4, 2)$, $B(-1, -3)$, $C(6, 2)$, and $D(x, -4)$.

8. Find the missing coordinate so that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ for $A(5, -3)$, $B(-2, 4)$, $C(3, 3)$, and $D(-7, x)$.

9. Write an equation of a circle with center at $(2, 3)$ and radius 6.

10. Write the equation of a circle with center at the origin which passes through the point $(-5, 12)$.

11. **Given:** Isosceles triangle ABC with vertices $C(b, c)$, $A(0, 0)$, $B(2b, 0)$ and midpoints E and D of $\overline{AC}$ and $\overline{BC}$ respectively.

    Prove: $AD = BE$

12. **Given:** Figure $ABCD$ with vertices $A(-b, 0)$, $B(0, a)$, $C(b, 0)$, and $D(0, -a)$

    Prove: $ABCD$ is a rhombus.

Cumulative Review (Chapters 11–14)

1. Find the area of a square inscribed in a circle with a diameter of 10 cm.
2. Find the area of an isosceles triangle with a base of 8 cm and each of the equal sides being 5 cm.
3. Find the area of an equilateral triangle if its perimeter is 12 cm.
4. If the ratio of the perimeter of two similar triangles is 2 : 1, what is the ratio of their areas?
5. Find the area of a regular hexagon inscribed in a circle with a radius of 6 cm.
6. If a prism has an altitude of 6 m and a base with an area of 12 m^2, what is its volume?
7. If a pyramid has a base that is a square 3 m on a side and an altitude of 7 m, what is the volume of the pyramid?
8. Find the surface area of a right circular cylinder if its height is 10 cm and its base has a diameter of 4 cm. (Use $\pi = 3.14$.)
9. Find the volume of the cylinder given in exercise 8.
10. If a triangle ABC with vertices $A(3, 5)$, $B(5, -2)$, and $C(-4, 3)$ were reflected about the y-axis, what would be the coordinates of its image?
11. If a triangle ABC with vertices $A(4, -2)$, $B(-3, -7)$, and $C(1, 6)$ were reflected about the x-axis, what would be the coordinates of its image?
12. Give the equations of the lines of symmetry of the square with vertices $(1, 1)$, $(5, 1)$, $(5, 5)$, and $(1, 5)$.
13. If the point $(2, 6)$ is rotated 180° about the origin, what would be the coordinates of its image?
14. Find the length of the hypotenuse of the right triangle with vertices at $(6, 8)$, $(9, -4)$, and $(1, -6)$.
15. Find the equation of a line that contains the points $(2, -6)$ and $(-4, 3)$.

Symbols

$\overleftrightarrow{AB}$	line AB (page 12)
$\parallel$	is parallel to (page 13)
$\overline{AB}$	segment AB (page 16)
$\overrightarrow{AB}$	ray AB (page 16)
$\angle ABC$	angle ABC (page 17)
$\triangle ABC$	triangle ABC (page 17)
$\odot O$	circle O (page 17)
AB	length of segment AB (page 20)
$\cong$	is congruent to (page 20)
$m\angle ABC$	measure of $\angle ABC$ (page 20)
$\not\cong$	is not congruent to (page 22)
$\perp$	is perpendicular to (page 28)
$p \rightarrow q$	p implies q (page 56)
$\sim p$	not p (page 60)
$p \leftrightarrow q$	p if and only if q (page 61)
$\nparallel$	is not parallel to (page 175)
$\not\perp$	is not perpendicular to (page 160)
$\neq$	is not equal to (page 161)
$\sqrt{}$	square root (page 227)
$\sim$	is similar to (page 313)
$\doteq$	is approximately equal to (page 331)
$\overset{\frown}{AB}$	minor arc determined by A and B (page 346)
$\overset{\frown}{ACB}$	major arc determined by A and B (page 346)
$m\overset{\frown}{AB}$	measure of $\overset{\frown}{AB}$ (page 346)
$A(R)$	area of region R (page 394)
$P(x, y)$	point P with coordinates x and y (page 505)

Table of Squares and Square Roots

N	N^2	$\sqrt{N}$	N	N^2	$\sqrt{N}$	N	N^2	$\sqrt{N}$	N	N^2	$\sqrt{N}$
1	1	1.000	26	676	5.099	51	2,601	7.141	76	5,776	8.718
2	4	1.414	27	729	5.196	52	2,704	7.211	77	5,929	8.775
3	9	1.732	28	784	5.292	53	2,809	7.280	78	6,084	8.832
4	16	2.000	29	841	5.385	54	2,916	7.348	79	6,241	8.888
5	25	2.236	30	900	5.477	55	3,025	7.416	80	6,400	8.944
6	36	2.449	31	961	5.568	56	3,136	7.483	81	6,561	9.000
7	49	2.646	32	1,024	5.657	57	3,249	7.550	82	6,724	9.055
8	64	2.828	33	1,089	5.745	58	3,364	7.616	83	6,889	9.110
9	81	3.000	34	1,156	5.831	59	3,481	7.681	84	7,056	9.165
10	100	3.162	35	1,225	5.916	60	3,600	7.746	85	7,225	9.220
11	121	3.317	36	1,296	6.000	61	3,721	7.810	86	7,396	9.274
12	144	3.464	37	1,369	6.083	62	3,844	7.874	87	7,569	9.327
13	169	3.606	38	1,444	6.164	63	3,969	7.937	88	7,744	9.381
14	196	3.742	39	1,521	6.245	64	4,096	8.000	89	7,921	9.434
15	225	3.873	40	1,600	6.325	65	4,225	8.062	90	8,100	9.487
16	256	4.000	41	1,681	6.403	66	4,356	8.124	91	8,281	9.539
17	289	4.123	42	1,764	6.481	67	4,489	8.185	92	8,464	9.592
18	324	4.243	43	1,849	6.557	68	4,624	8.246	93	8,649	9.644
19	361	4.359	44	1,936	6.633	69	4,761	8.307	94	8,836	9.695
20	400	4.472	45	2,025	6.708	70	4,900	8.367	95	9,025	9.747
21	441	4.583	46	2,116	6.782	71	5,041	8.426	96	9,216	9.798
22	484	4.690	47	2,209	6.856	72	5,184	8.485	97	9,409	9.849
23	529	4.796	48	2,304	6.928	73	5,329	8.544	98	9,604	9.899
24	576	4.899	49	2,401	7.000	74	5,476	8.602	99	9,801	9.950
25	625	5.000	50	2,500	7.071	75	5,625	8.660	100	10,000	10.000